Realization of a Constant Phase Difference

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This paper bears on the problem of splitting a signal into two parts of like amplitudes but different phases. Constant phase differences are utilized in such circuits as Hartley single sideband modulators. The networks considered here are pairs of constant-resistance phase-shifting networks connected in parallel at one end. The first part of the paper shows how to compute the best approximation to a constant phase difference obtainable over a prescribed frequency range with a network of prescribed complexity. The latter part shows how to design networks producing the best approximation.

A PERENNIAL problem is that of designing a circuit to split a signal into two parts which are the same in amplitude but which differ in phase by a constant amount. A 90-degree phase difference is needed, for example, in the single sideband modulation system due to R. V. L. Hartley.[1] It is well known that it is not possible to obtain exactly equal amplitudes and exactly constant phase differences at all frequencies except in the trivial special case of a 180-degree phase difference. Various methods have been devised, however, for approximating these characteristics over finite frequency ranges. The most obvious method is to use a pair of constant resistance phase shifting sections in parallel at one end and with separate terminations at the other end[2] as indicated in Fig. 1.

This paper is devoted to the problem of obtaining approximately constant phase differences under the specific assumption that pairs of constant resistance phase shifting networks are to be used. The paper has been written with two objects in mind. The first is the development of a method for determining the best approximation to a constant phase difference which can be obtained over a prescribed frequency range with a pair of phase shifting networks of a prescribed total complexity. The second object is the description of a straightforward design procedure by means of which the networks can be designed to give this best possible approximation.

The problem under consideration is typical of those usually described as problems in network synthesis. In other words, a network of a prescribed general type is to be designed to approximate as closely as possible an ideal operating characteristic of a prescribed form. The same procedure will be followed as that appropriate for most such problems. The procedure begins with the development of a mathematical expression representing the most

[1] *U. S. Patent* 1,666,206, 4/17/28, Modulation System.
[2] Another common method uses reactance shunt branches between effectively infinite impedances, such as the plate and grid impedances of screen grid tubes.

general characteristics which can be obtained with the prescribed type of network. This is followed by the determination of particular choices of the arbitrary constants in the expression, which will lead to the best approximation to the prescribed ideal characteristic. The next step is to determine formulae for the degree of approximation to the ideal, which will be

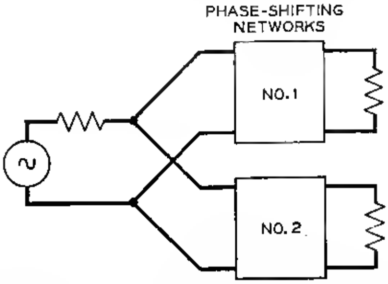

Fig. 1—Phase-shifting networks for approximation to a constant phase difference.

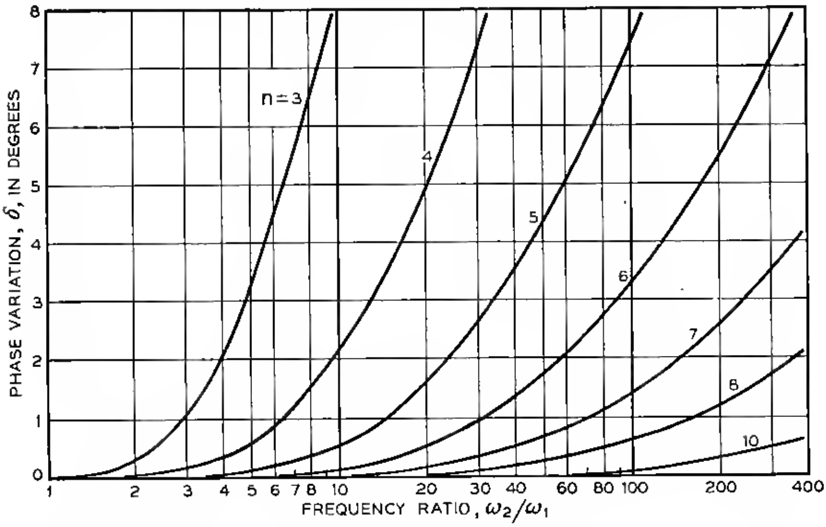

Fig. 2—Variation in phase difference, when average is 90°, with a network of n sections.

obtained with those particular values of the constants. The final step is the development of a method for determining corresponding actual networks.

From the optimum choice of constants, curves can be calculated which show what can be done with a network of any given complexity (Fig. 2). Then the complexity needed for any particular application can be read directly from the curves. The special choice of constants also leads to special

formulae for element values of corresponding networks, using tandem sections of the simplest all-pass type (Fig. 3).

Form of the $\tan\left(\frac{\beta}{2}\right)$ Function

If β_1 and β_2 represent the phase shifts through the two constant resistance networks of Fig. 1, then $\tan\left(\frac{\beta_1}{2}\right)$ and $\tan\left(\frac{\beta_2}{2}\right)$ must both be realizahle as the reactances of physical reactance networks. In other words, these quantities must be odd rational functions of ω witb real coefficients and must also meet various other special restrictions. If β is used to represent the phase difference $\beta_2 - \beta_1$, the function $\tan\left(\frac{\beta}{2}\right)$ must also be an odd rational function of ω with real coefficients. Because of the minus sign

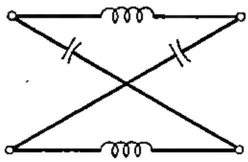

Fig. 3—Simplest all-pass section.

associated with β_1 in the definition of β, however, $\tan\left(\frac{\beta}{2}\right)$ does not have to meet the additional restrictions which must be imposed upon $\tan\left(\frac{\beta_1}{2}\right)$ and $\tan\left(\frac{\beta_2}{2}\right)$. In a later part of the paper a method will be described by which a pair of physical phase shifting networks can be designed to produce any $\tan\left(\frac{\beta}{2}\right)$ function which is an odd rational function of ω with real coefficients.

In any range where the phase difference β approximates a constant, the function $\tan\left(\frac{\beta}{2}\right)$ will also approximate a constant. Hence, the present problem is really that of approximating a constant over a given frequency range with an odd rational function of ω with real coefficients. In this problem, the degree of the function must be assumed to be prescribed as well as the frequency range in which a good approximation is to be ohtained, for the degree of the function determines the complexity of the corresponding network.

W. Cauer shows how functions of certain types can be designed to approx-

imate unity in prescribed frequency ranges.[3] These functions, however, are not odd rational functions of frequency but are irrational functions appropriate to represent filter image impedances or the hyperbolic tangents or cotangents of filter transfer constants. It turns out, however, that they can be transformed into odd rational functions of the desired type by a simple transformation of the variable.

Each of Cauer's functions is said to approximate a constant in tbe Tchebycheff sense, which means that in the prescribed range of good approximation the maximum departure from the approximated constant is as small as is permitted by the specifications on the frequency range and the degree of the function. Each function also has the property of exhibiting series of equal maxima and equal minima in the range of good approximation, such as those indicated in the illustrative β curve[4] of Fig. 4.

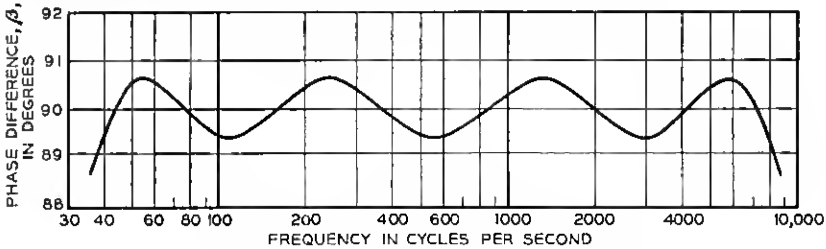

Fig. 4—Example of a phase difference characteristic.

Of the various forms in which Cauer's Tchebycheff functions F can be expressed, the following form is the one appropriate for showing how odd rational functions of frequency can be obtained:

$$
(1)\quad \begin{cases}
\text{When } n \text{ is odd} \\[4pt]
F = U\sqrt{1 - X^2}\displaystyle\prod_{s=1}^{s=\frac{n-1}{2}} \frac{\left[1 - sn^2\left(\frac{2s-1}{n}K, k\right)X^2\right]}{\left[1 - sn^2\left(\frac{2s}{n}K, k\right)X^2\right]}. \\[8pt]
\text{When } n \text{ is even} \\[4pt]
F = \dfrac{U}{\sqrt{1 - X^2}} \dfrac{\displaystyle\prod_{s=1}^{s=\frac{n}{2}}\left[1 - sn^2\left(\frac{2s-1}{n}K, k\right)X^2\right]}{\displaystyle\prod_{s=1}^{s=\frac{n}{2}-1}\left[1 - sn^2\left(\frac{2s}{n}K, k\right)X^2\right]}.
\end{cases}
$$

[3] "Ein Interpolationsproblem mit Funktionen mit Positivem Realteil," *Mathematische Zeitschrift*, 38, 1–44 (1933).

[4] The data for the illustrative curve were obtained from a trial design carried out by P. W. Rounds.

In these equations, the symbol *sn* indicates an elliptic sine, of modulus k, while K represents the corresponding complete elliptic integral. U is merely a constant scale factor, while n is an integer measuring the complexity of corresponding networks. In the case of phase-difference networks, n represents the total number of sections of the type indicated in Fig. 3, which are included in the two phase-shifting networks or their tandem section equivalents.

In Cauer's filter theory, the variable X represents a rational function of ω which permits F to be an image impedance or a $\coth\left(\frac{\theta}{2}\right)$ function. In order that F may be an odd rational function of ω, however, as is required when it is to represent $\tan\left(\frac{\beta}{2}\right)$, X must be defined by the relation

$$\omega = \omega_2\sqrt{1 - X^2}. \tag{2}$$

Cauer shows that F approximates a constant in the Tchebycheff sense in the range $0 < X < k$. Hence, in terms of ω, the range of approximation is $\omega_1 < \omega < \omega_2$, where ω_1 and ω_2 are arbitrary provided the modulus k is assumed to be determined by the relation

$$k = \frac{\sqrt{\omega_2^2 - \omega_1^2}}{\omega_2}. \tag{3}$$

Alternative Expression for the $\tan\left(\frac{\beta}{2}\right)$ Function

While equations (1) are the most convenient form of F to use in deriving the transformation of the variable, an alternative more compact form is more suitable for determining the degree of approximation to a constant phase difference and the element values of corresponding networks. When F represents $\tan\left(\frac{\beta}{2}\right)$ and hence ω and X are related as in (2), the equivalent expression is as follows:[5]

$$\begin{aligned} \tan\left(\frac{\beta}{2}\right) &= U\, dn\left(nu\frac{K_1}{K}, k_1\right) \\ \omega &= \omega_2\, dn(u, k). \end{aligned} \tag{4}$$

In this expression, *dn* represents a so-called "*dn*" function, the third type of Jacobian elliptic function usually associated with the elliptic sine, or *sn* function, and the elliptic cosine, or *cn* function. The symbol u represents

[5] This expression depends on a so called modular transformation of elliptic functions not found in the usual elliptic function text. The transformation theory may be found in "An Elementary Treatise on Elliptic Functions," Arthur Cayley, G. Bell & Sons, London, 1895.

a "parametric variahle" which would be eliminated on forming a single equation from the two simultaneous equations indicated. The modulus k_1, of the *dn* function corresponding to $\tan\left(\frac{\beta}{2}\right)$ is related to the modulus k, of the *dn* function corresponding to ω, in the manner indicated helow. The constant K_1, of course, represents the complete integral of modulus k_1, just as K represents the complete integral of modulus k.

Corresponding to any modulus k there is a so-called modular constant q. Using q_1 to represent the corresponding modular constant of modulus k_1, it is here required that

$$q_1 = q^n. \tag{5}$$

One modulus can be computed from the other by means of this relationship and tahulations of $\log_{10} q$ vs $\sin^{-1} k$ which are included in most elliptic function tables.[6]

Degree of Approximation to a Constant Phase Difference

When u is real and varies from zero to infinity, the corresponding value of ω as determined hy (4) merely oscillates hack and forth hetween the values ω_1 and ω_2. In other words, it merely crosses hack and forth across the range in which $\tan\left(\frac{\beta}{2}\right)$ approximates a constant. Similarly, when u is real and increases from zero to infinity, $\tan\left(\frac{\beta}{2}\right)$ oscillates between $U\sqrt{1 - k_1^2}$ and U. The *equal ripple* property of the curve illustrated in Fig. 4 is explained hy the fact that tbe period of oscillation of $\tan\left(\frac{\beta}{2}\right)$ with respect to u is merely a fraction of that of ω, so that $\tan\left(\frac{\beta}{2}\right)$ passes through several ripples while the value of ω moves from ω_1 to ω_2.

Comhining the formulae for the maximum and minimum values of $\tan\left(\frac{\beta}{2}\right)$ gives the relation

$$\tan\left(\frac{\delta}{2}\right) = \frac{U(1 - \sqrt{1 - k_1^2})}{1 + U^2\sqrt{1 - k_1^2}} \tag{6}$$

[6] When k is extremely close to unity, it may be easier to obtain accurate computations by using the additional relation

$$\log_{10}(q)\log_{10}(q') = \left(\frac{\pi}{\log_e(10)}\right)^2$$

where q' is the modular constant of modulus $\sqrt{1 - k^2} = \frac{\omega_1}{\omega_2}$.

in which δ represents the total variation of the phase difference β in the approximation range. Similarly, the average value β_a of β in the approximation range is given by[7]

$$\tan(\beta_a) = \frac{U(1 + \sqrt{1 - k_1^2})}{1 - U^2\sqrt{1 - k_1^2}}. \tag{7}$$

If the phase variation δ is reasonably small, (6) and (7) can be replaced by the approximate relationships

$$\begin{aligned} \delta &= \frac{\sin(\beta_a)}{2} k_1^2 \quad \text{radians} \\ \tan\left(\frac{\beta_a}{2}\right) &= U\sqrt[4]{1 - k_1^2}.^{8} \end{aligned} \tag{8}$$

A still further modification is obtained by replacing k_1^2 by the quantity $16q_1$, which is an approximate equivalent when k_1^2 is small, and by then replacing q_1 by the equivalent q^n of (5). This gives

$$\delta = 8\sin(\beta_a)q^n \tag{9}$$

$$\tan\left(\frac{\beta_a}{2}\right) = U\sqrt[4]{1 - 16q^n}.$$

When combined with (3) and tabulations of $\sin^{-1}(k)$ vs $\log_{10}(q)$, these formulae can be used to compute δ when the parameters ω_1, ω_2, β_a and n are prescribed. Curves of δ are plotted against ω_2/ω_1 in Fig. 2, assuming β_a to be 90 degrees.

Determination of a Network Corresponding to a General Phase Difference Function

Since $\tan\left(\frac{\beta}{2}\right)$ must be an odd rational function of ω, it can be expressed in the form

$$\tan\left(\frac{\beta}{2}\right) = \frac{\omega B}{A} \tag{10}$$

in which A and B are even polynomials in ω. This requires

$$\frac{\beta}{2} = \arg(A + i\omega B). \tag{11}$$

[7] More exactly, β_a is the average of the maximum and minimum values of β occurring in the range of approximation.

[8] In the important special case in which the average phase difference β_a is 90°, this expression for $\tan\left(\frac{\beta_a}{2}\right)$ is exact rather than approximate.

Similarly, if attention is focused on the phase shifts of the individual phase-shifting networks rather than on the phase difference, the following odd rational functions can be introduced:

(12)
$$\tan\left(\frac{\beta_1}{2}\right) = \frac{\omega B_1}{A_1}$$
$$\tan\left(\frac{\beta_2}{2}\right) = \frac{\omega B_2}{A_2}$$

in which A_1, B_1, A_2, and B_2 are additional even polynomials in ω. This requires

(13)
$$\frac{\beta_1}{2} = \arg\,(A_1 + i\omega B_1)$$
$$\frac{\beta_2}{2} = \arg\,(A_2 + i\omega\beta_2).$$

It also requires

(14) $$-\frac{\beta_1}{2} = \arg\,(A_1 - i\omega B_1).$$

Since the argument of a product is the sum of the arguments of the separate factors, (13) and (14) require

(15) $$\frac{\beta}{2} = \frac{\beta_2 - \beta_1}{2} = \arg\,(A_2 + i\omega B_2)(A_1 - i\omega B_1).$$

This permits us to write

(16) $$(A_2 + i\omega B_2)(A_1 - i\omega B_1) = H(A + i\omega B)$$

in which H is a real constant.

When $\tan\left(\frac{\beta}{2}\right)$ is prescribed, a corresponding polynomial of the form $(A + i\omega B)$ can readily be derived. The problem is then to factor it into the product of two polynomials $(A_2 + i\omega B_2)$ and $(A_1 - i\omega B_1)$ such that A_1, B_1, A_2, and B_2 determine physically realizable phase shifts through (12). Two factors of the general form $(A_2 + i\omega B_2)$ and $(A_1 - i\omega B_1)$ can readily be obtained in a number of ways. The only question is how to obtain them in such a way that the corresponding phase characteristics will be physical. A procedure meeting this requirement is described below.

The variable ω is first replaced in $(A + i\omega B)$ by p representing $i\omega$. This leaves a polynomial in p with real coefficients, since A and B represent polynomials in ω^2, while p^2 represents $-\omega^2$. Suppose all the roots of the polynomial $A + pB$ are determined. Then this polynomial can be split into

two factors by assigning various of the roots to each of the two factors. It turns out that physically realizable phase characteristics will be obtained if all those roots with positive real parts are assigned to the factor $(A_1 - pB_1)$ which appears in (16) when $i\omega$ is replaced by p, all other roots being assigned to the factor $(A_2 + pB_2)$.

The physical realizability of the above division of the roots follows from a theorem which states that $\frac{pB_x}{A_x}$ is realizable as the impedance of a two-terminal reactance network whenever A_x and B_x are even polynomials in p with real coefficients such that $A_x + pB_x$ has no roots with positive real parts.[9] From this theorem and the fact that the evenness of A_x and B_x causes them to remain unchanged when p is reversed in sign, it follows that $\frac{pB_x}{A_x}$ will also be the impedance of a physical two-terminal reactance network whenever $A_x - pB_x$ has no roots with *negative* real parts. Thus, by (12) the above division of the roots of $A + pB$ makes $\tan\left(\frac{\beta_1}{2}\right)$ and $\tan\left(\frac{\beta_2}{2}\right)$ realizable as the impedances of two-terminal reactance networks. These reactance networks and their inverses are merely the arms of unit impedance lattices producing the phase characteristics defined by (12).

The above argument merely shows that each of the two phase-shifting networks can at least be realized as a single lattice when $\tan\left(\frac{\beta_1}{2}\right)$ and $\tan\left(\frac{\beta_2}{2}\right)$ are determined by the method described. Actually, they can be broken into tandem sections directly as soon as the roots of $(A_1 - pB_1)$ and $(A_2 + pB_2)$ have been determined. From $(A_1 - pB_1)$, the quantity $(A_1 + pB_1)$ can be found by merely reversing the signs of the roots. Then by using the principle that the argument of a product is the sum of the arguments of the separate factors, phase-shifting networks can be designed corresponding to various factors or groups of factors as determined from the known roots of $(A_1 + pB_1)$ and $(A_2 + pB_2)$. There can be a separate section for each real root and each conjugate pair of complex roots.[10]

Determination of a Network Corresponding to a Tchebycheff Type of Phase Difference Characteristic

The procedure described above for determining a network corresponding to a general phase difference characteristic is complicated by the necessity

[9] See "Synthesis of Reactance 4-Poles which Produce Prescribed Insertion Loss Characteristics," *Journal of Mathematics and Physics*, Vol. XVIII, No. 4, September, 1939—page 276.

[10] See H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Company, New York, 1945, Page 239, §11.6.

of determining the roots of the polynomial $A + pB$. In the case of the Tchebycheff type of characteristic described in the first part of the paper, the required roots can be determined by means of special relationships. In the first place, the roots of $A + pB$ are the roots of $\left(1 + i \tan\frac{\beta}{2}\right)$. In other words, by equation (4) they are the roots of $\left[1 + iU\, dn\left(nu\frac{K_1}{K}, k_1\right)\right]$. The values of u at the roots turn out to have an imaginary part iK', where K' is the complete elliptic integral of modulus $\sqrt{1 - k^2}$. If a new variable u' is defined by

$$u = u' + iK' \tag{17}$$

the roots can be shown to correspond to the values of u' determined by

$$\frac{sn\left(nu'\frac{K_1}{K}, k_1\right)}{cn\left(nu'\frac{K_1}{K}, k_1\right)} = -U. \tag{18}$$

If it is assumed that the phase variation is small in the range of approximation to a constant, it can be shown that one value of u' determined by the above relation is given approximately by

$$\frac{nu'\pi}{K} = -\beta_a \tag{19}$$

where β_a is the average phase difference for the range of approximation as before (in radians). After this value of u' has been computed, all the roots of $\left[1 + iU\, dn\left(nu\frac{K_1}{K}, k_1\right)\right]$ can be found by computing the values of ω corresponding to this value of u' and to those values obtained by adding integral multiples of the real period $\frac{2K}{n}$ of $dn\left(nu\frac{K_1}{K}, k_1\right)$. This gives the following formula for the roots in terms of $p = i\omega$.

$$p_\sigma = \omega_2 \frac{cn\left(\frac{2\sigma K}{n} + u_0'\right)}{sn\left(\frac{2\sigma K}{n} + u_0'\right)}, \qquad \sigma = 0, \cdots, (n-1) \tag{20}$$

in which u_0' is the value of u' determined by (19).

Finally, instead of using the above elliptic function formula directly, one may replace the elliptic functions by equivalent ratios of Fourier series expansions of θ functions. This gives

$$p_\sigma = \sqrt{\omega_1\omega_2}\,\frac{\cos(\lambda_\sigma) + q^2\cos(3\lambda_\sigma) + q^6\cos(5\lambda_\sigma)\cdots}{\sin(\lambda_\sigma) - q^2\sin(3\lambda_\sigma) + q^6\sin(5\lambda_\sigma)\cdots} \tag{21}$$

in which the angle λ_σ is defined by

$$\lambda_\sigma = \frac{\sigma \cdot 180^\circ - \frac{1}{2}\beta_a}{n} \text{ degrees}, \qquad \sigma = 0, \cdots, (n-1). \tag{22}$$

Because all the p_σ's are real in this Tchebycheff case, corresponding networks can be made up of sections of the simple type indicated in Fig. 3. In one of the two phase-shifting networks there will be one section for each positive p_σ, and it will be given by

$$L = \frac{R_0}{p_\sigma} \qquad C = \frac{1}{R_0 p_\sigma}$$

where R_0 is the image impedance. Similarly, in the second phase-shifting network there will be one section for each negative p_σ, and it will be given by

$$L = -\frac{R_0}{p_\sigma} \qquad C = \frac{-1}{R_0 p_\sigma}.$$